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The midrange wavenumber spectrum of van Gogh's *Starry Night* does not obey a turbulent inertial range scaling law

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ABSTRACT

The swirling night sky painted by van Gogh in his *Starry Night* painting has been interpreted as a representation of turbulence. Indeed, a popular myth is that this painting quantitatively (albeit coincidentally) captures the scaling behaviour of actual turbulence. One of the most familiar scaling laws in turbulence is the Kolmogorov scaling law, whereby the two-point autocorrelation of the velocity field obeys a −5/3 power law dependence in the midrange ('inertial range') wavenumbers. Compressible turbulence also obeys a similar power law, but with an exponent between −5/3 and −2, depending on Mach number. An examination of the spectral dependence of a two-point autocorrelation of the image intensity in midrange wavenumbers shows that van Gogh's *Starry Night* does obey a power law, but instead with an exponent of −1. Examination of various other van Gogh and impressionist artists' paintings also do not exhibit the correct power law exponent for a turbulent flow. Thus, van Gogh's *Starry Night* painting does not accurately embody one of the most fundamental features of turbulent flows, weakening the foundations of the popular and commonly reiterated myth that this painting captures turbulence in a mathematically correct way.

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Introduction

Spatial wavenumber spectra of the image intensity of many photographic images and paintings of natural scenes typically exhibit a power law dependence in their midrange wavenumbers [1]. Although the value of the power law exponent is different for different images, given the large number of paintings that exist, it is possible that some paintings that depict a turbulent flow may coincidentally exhibit spectra with a −5/3 midrange exponent. Such spatial dependence is precisely what occurs in actual turbulent flow spectra, a feature well known to turbulence researchers as the Kolmogorov scaling law. In this vein, previous authors have indeed found similarities in the shape of the probability density distribution of two point autocorrelations in van Gogh's famous *Starry Night* painting when compared with incompressible turbulent flows [2], leading to many news media articles and websites stating that this painting quantitatively captures the physics of incompressible turbulent flow. However, a more satisfying confirmation that van Gogh's *Starry Night*

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painting accurately mimics turbulence would be if the midrange wavenumber dependence of a two-point correlation from this painting obeyed Kolmogorov's scaling law. Unpublished work [3] finds that the midrange wavenumber zone of a two-point autocorrelation of *Starry Night* exhibits a power law scaling with $n = -2$ like that seen in compressible turbulent star forming regions, giving renewed life to the myth that *Starry Night* mathematically mimics turbulent flows. However, an error is present in [3], which prompted the present reexamination of the wavenumber dependence of this painting.

Methods

A principal characteristic of high Reynolds number equilibrium turbulent flows is their midrange ('inertial zone') wavenumber dependence on spatial wavenumber k. In particular, defining the two point correlation

$$R_{ij}(\boldsymbol{r}) = \int v_i(\boldsymbol{x}) v_j(\boldsymbol{x} + \boldsymbol{r}) d\boldsymbol{x} \tag{1}$$

where, $v_i(\boldsymbol{x})$ is ith cartesian component of the velocity field at position $\boldsymbol{x}$, and also defining the Fourier coefficients of this correlation as

$$\hat{R}_{ij}(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int R_{ij}(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r} \tag{2}$$

then a commonly defined one-dimensional wavenumber spectrum is obtained by integrating over two of the wavenumber directions, giving

$$E_{11}^{(1)}(k_1) = \iint \hat{R}_{11}(\boldsymbol{k}) dk_2 dk_3 \tag{3}$$

For incompressible flows, experimental measurements of $E_{11}^{(1)}$ exhibit a particular power law dependence (see e.g. [4]) in the inertial range given by

$$E_{11}^{(1)} \propto k_1^{-5/3} \tag{4}$$

This $-5/3$ power law spectral behaviour is an essential feature of high Reynolds number incompressible turbulence. Dimensional arguments were first used by Kolmogorov to explain the value $n = -5/3$ for the power law exponent, and equation (4) is commonly referred to as the Kolmogorov scaling law. For compressible flows, a power law dependence is still seen in the inertial range wavenumber spectrum, but the power n increases with Mach number Ma from $n = -5/3$ for $Ma \leq 1$ to $n = -2$ as $Ma \to \infty$ [5].

In examining whether van Gogh's *Starry Night* quantitatively mimics turbulent scaling laws, previous authors [2,3] have considered the second order structure function

$$SF_2(\boldsymbol{r}) = \int (I(\boldsymbol{x} + \boldsymbol{r}) - I(\boldsymbol{r}))^2 d\boldsymbol{r} \tag{5}$$

where I is some measure of pixel image intensity (e.g. gray/red/green/blue channel level, or luminance). While it is possible to proceed by obtaining SF_2 via numerical evaluation of this integral over all pixels in the image for all pixel separations $\boldsymbol{r}$ in the image, an easier approach exists. In particular, it can be shown that SF_2 in a turbulent flow has the same inertial range power law wavenumber dependence as the more commonly measured two-point correlation given in equation (1). Thus, here we instead consider the image intensity version of (1):

$$I_{11}(\boldsymbol{r}) = \int I(\boldsymbol{x})I(\boldsymbol{x}+\boldsymbol{r})d\boldsymbol{x} \tag{6}$$

and its Fourier coefficients $\hat{I}_{11}(\boldsymbol{k})$. Equation (6) is a form of convolution integral, so that the Fourier coefficients of $I_{11}(\boldsymbol{r})$ can be obtained by simple multiplication in Fourier space. The midrange wavenumber power law exponent of a one-dimensional spectrum obtained as in equation (3) can thus be obtained by examining the square of the magnitude of the Fourier coefficient of the image intensity $I(\boldsymbol{r})$, i.e. by examining

$$|\hat{I}_{11}(\boldsymbol{k})| = |\hat{I}(\boldsymbol{k})|^2 \tag{7}$$

For a two-dimensional image, the midrange power law exponent of the image intensity spectrum equivalent to the one-dimensional spectrum $E_{11}^{(1)}(k_1)$ is then easily obtained by examining the sum of the squares over one of the two wavenumber directions of the discrete fast Fourier transform coefficients. Denoting the resulting one-dimensional spectrum as $I_{11}^{(1)}(k_1)$, MATLAB's image input and fast Fourier transform (FFT) commands were used on the pixel levels of a variety of two-dimensional discrete images to obtain $I_{11}^{(1)}(k_1)$. Straight lines associated with various power law exponents where then overlaid on log–log plots of $I_{11}^{(1)}(k_1)$ versus k_1 to find the resulting midrange wavenumber power law scaling of different images.

Results

The one-dimensional spectrum of a 1114×680 pixel image of a sky-containing portion of van Gogh's *Starry Night* painting is shown in Figure 1. Grey scale pixel values were used for image intensity I in Figure 1. Using either red channel (R) pixel levels or luminance ($0.299\text{R} + 0.587\text{G} + 0.114\text{B}$, used in [2]), or considering a 1920×1080 pixel image of the entire Starry Night painting, did not change the midrange power law dependence $n = -1$ seen in Figure 1.

The midrange power law exponents n determined in the above manner for a variety of other images are listed in Table 1. All of van Gogh's paintings examined here exhibited power law exponents less negative than those seen with turbulent flows. The same is true of two other impressionist painters, Monet and Morisot, whose images are considered in Table 1. In contrast, Table 1 shows that when an image of the velocity field of a direct numerical simulation of an incompressible flow was analysed using the present method, the correct $n = -5/3$ power law scaling was observed in the inertial range. Similarly, the last row of Table 1 shows that when an image from a simulation of a high Mach number compressible flow in an astrophysical gas cloud was analysed, the present approach gave the correct $n = -2$ power law dependence.

Discussion

The spectral dependence of a two-point autocorrelation $I_{11}^{(1)}(k_1)$ obtained from van Gogh's *Starry Night* painting does not obey the power law midrange wavenumber behaviour of

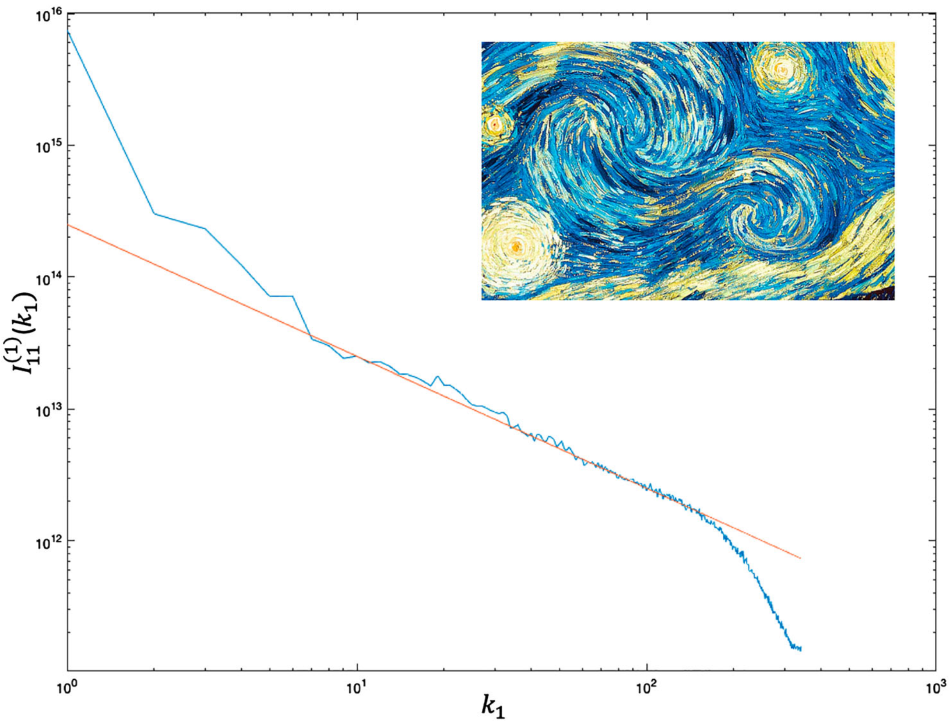

Figure 1. The one-dimensional spectrum $I_{11}^{(1)}(k_1)$ of a clipped portion containing the sky in van Gogh's *Starry Night* painting, seen inset at upper right below, is shown versus wavenumber for the two-point autocorrelation of image intensity. The straight line shows a power law dependence with exponent $n = -1$. The portion of *Starry Night* shown was obtained by clipping the open source image at http://wallpaperswide.com/download/the_starry_night-wallpaper-1920×1080.jpg.

Table 1. Values of the power law exponent for the midrange wavenumber range of the two-point autocorrelation $I_{11}^{(1)}(k_1)$ are listed for a variety of images.

Image	Power law exponent n
van Gogh *Starry Night*	−1
van Gogh *Starry Night Over the Rhone*	−1
van Gogh *Café Terrace at Night*	−1
van Gogh *The White House at Night*	−1.5
van Gogh *Sunflowers*	−1
van Gogh *Irises*	−1.25
van Gogh *Self Portrait* 1889	−1.5
Monet *Water Lilies*	−1.2
Monet *Impression, Soleil Levant*	−1.2
Monisot *Summer's Day*	−1.35
Image of turbulent velocity field from [6]	−5/3
van Gogh *Starry Night* image with sky portion replaced using data from simulated compressible giant molecular cloud turbulence [7]	−2

a turbulent flow. Instead, here we find a k^{-1} wavenumber dependence for the midrange wavenumbers. While reference [3] finds *Starry Night* has a k^{-2} two-point autocorrelation spectral dependence like that seen in high Mach number compressible turbulence, they obtained $SF_2(\boldsymbol{r})$ in equation (5) directly and then mistakenly considered the square of its

Fourier coefficients, rather than the amplitude of its Fourier coefficients. Thus, correcting their error by taking the square root of k^{-2}, their results would have agreed with the present finding of a k^{-1} midrange wavenumber dependence.

While *Starry Night* does not exhibit the characteristic spectral dependence that is seen in turbulent flows, Aragón et al. [2] find that probability distribution functions (PDFs) of $\delta v = I(\boldsymbol{x} + \boldsymbol{r}) - I(\boldsymbol{r})$ have similarities with PDFs of turbulent flow velocity fields. However, the fact that *Starry Night* does not embody the wavenumber dependence of turbulent flows, while images of these flows do exhibit the correct dependence, weakens the claim that this painting mimics the physics of a turbulent flow.

Besides the above noted lack of quantitative agreement between turbulent flows and the swirling sky depicted in *Starry Night*, previous arguments that pixel levels in this painting obey turbulent scaling laws require an assumption that image intensity is analogous to flow velocity. To this author's eye, image intensity in *Starry Night* does not correlate with flow speed. This further diminishes the validity of the above noted myth. While others may interpret this painting differently, regardless of interpretation, this painting does not obey one of the primary features of turbulent flows i.e. the midrange wavenumber behaviour of one-dimensional spectra obtained from a two-point autocorrelation of this painting decays too slowly to represent a turbulent flow in a mathematically rigorous manner.

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Disclosure statement

No potential conflict of interest was reported by the author.

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